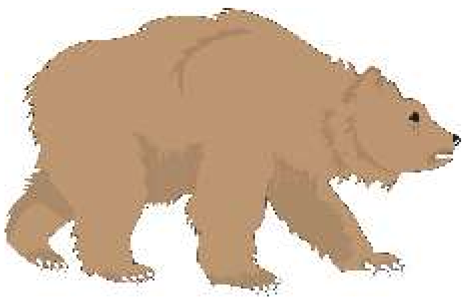
Philosophical Communications, Web Series, No. 32, pp. 187-207.
Dept. of Philosophy, Göteborg University, Sweden
ISSN 1652-0459

Ursus Philosophicus

Essays dedicated to Björn Haglund on his sixtieth birthday



Perspectives on the dispute between intuitionistic and classical mathematics*

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Abstract

It is not unreasonable to think that the dispute between classical and intuitionistic mathematics might be unresolvable or 'faultless', in the sense of there being no objective way to settle it. If so, we would have a pretty case of *relativism*. In this note I argue, however, that there is in fact not even disagreement in any interesting sense, let alone a faultless one, in spite of appearances and claims to the contrary. A position I call *classical pluralism* is sketched, intended to provide a coherent methodological stance towards the issue. Some reasons to recommend this stance are given, as well as some speculations as to why not everyone might want to follow the recommendation.

1. Introduction

Characteristic of a relativist position is the occurrence of what Crispin Wright has called *faultless disagreement*: two parties disagree over a specific claim, but there is no objective way to settle the dispute; no objective perspective from which one party can be said to be more justified than the other. The non-relativist holds that faultless disagreement does not occur, or more strongly, that it cannot occur.

The chances of finding cases of faultless disagreement would appear to depend on the area disputed, being perhaps greater for matters of taste, or matters of morals, than for factual matters. In this note I look at a particular area – mathematics – and a particular dispute

* I have not yet had the opportunity to discuss this note with my colleagues. However, recent conversations with Per Martin-Löf and Peter Pagin on related matters have been very helpful. It is still more than likely that mistakes or infelicities remain. I apologize for these in advance. The work reported here was partially supported by a grant from the Bank of Sweden Tercentenary Foundation for the project *Relativism*.

– the one between classical and intuitionistic mathematics. Can one find instances of faultless disagreement here?

One might hesitate to call mathematical matters 'factual'. But the point is that – in contrast with matters of taste or morals – it is undisputed that the notions of *truth*, *knowledge*, and *justification* are central. Also, even if mathematics is not directly about the physical world, it can be *applied* to that world.

Prima facie this dispute may look like a promising candidate for faultless disagreement. There certainly appears to be *disagreement*, since intuitionists reject as unfounded or outright false parts of classical mathematics, and recommend a severe revision of its methods. The rejection is based on careful and sophisticated analysis of those methods. But it leads to much more than a replacement of certain pieces of mathematics by others. The entire classical notion of *truth* in mathematics is revised, as are the classical meanings of the most common expressions in the language of mathematics. The very foundations of mathematics are shaken. The case for classical mathematics, on the other hand, is based not so much on foundational analysis as on the overwhelming success that the vast majority of past and present mathematicians have had using classical methods, including those that intuitionists deem illicit. Classical mathematics dominates the scene, and apparently with good reason.

In view of the seemingly fundamental differences in outlook, and the ways in which the respective mathematical practices are motivated, one may indeed wonder if this dispute could be settled in an objective way. Perhaps the disagreement is faultless.

During the last century mathematicians and logicians not only developed intuitionistic mathematics to great lengths, but also studied in detail numerous partial axiomatizations and their exact relationship to axiomatizations of parts of classical mathematics. I have nothing to add in this respect. Indeed, some remarks below would probably strike experts as crude and oversimplified. But the viewpoint I am trying to take in this note is not that of the expert but rather of an outside observer, trying to understand what is going on.

There are many varieties of intuitionism and constructive mathematics. For my purposes it will suffice to roughly distinguish (a) Brouwerian intuitionism, in which choice sequences form an essential ingredient; (b) modern constructivism as formulated by Per Martin-Löf and others working in constructive type theory; (c) the philosophical or meaning-theoretic intuitionism of Michael Dummett, Dag Prawitz, and others.[1] But most of what I say is meant to apply to all of these; it will be noted where the distinction matters.

Thus, I am concerned with the question of a reasonable methodological stance towards the conflict – if there is one – between constructivist and classical mathematics. This question may be mildly interesting in the philosophy of mathematics, if only for the pedagogical purpose of clarifying the sometimes bewildering statements about the nature of this conflict that an outsider comes across. Most bewildering, to my mind, is the statement, emphasized in various forms by almost all intuitionist practitioners and many classical mathematicians as well, that certain intuitionistic results *contradict* classical ones. Contradictions are always a serious matter, one would think, but in mathematics a disaster.

It may be fruitful to frame the question in the terminology pertaining to philosophical discussions of relativism. And the question ought to interest someone concerned with the coherence or adequacy of relativism in general, since this particular dispute is, at least on the face of it, exceedingly clearcut, and yet a real, live one. In fact, it is not easy to find similar examples in other areas.

My conclusion, however, will be that not only is there no faultless disagreement between classical and intuitionistic mathematics; there is no real disagreement at all. I sketch a position which I call *classical pluralism* (in either a strong or a weak version). The argument is not meant to reveal deep insights. Still, spelling it out in explicit terms might serve to dispel some misunderstandings, or so I hope.

In section 2 below I sort out three forms of disagreement that seem relevant in the present context. Section 3 spells out the position of classical pluralism, first (3.1) in terms of the mathematical objects talked about, and then (3.2) in terms of the words used. At the end of section 3.2 I try to give a somewhat precise version of the claim that there really is no disagreement going on. In section 4, finally, I briefly raise the question of whether pluralism could be a constructive stance rather than a classical one.

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[1] The list is of course not exhaustive; one should mention at least the Russian constructivist school initiated by Markov, Bishop's intuitionism, and topos theory. For an overview, cf. Troelstra and van Dalen (1988).

2. Disagreement

There are numerous mathematical claims, expressed in a syntax used by both parties, to which classical mathematicians and intuitionists take radically different attitudes. The most well-known example is the *Principle of Excluded Middle*, say, in the context of elementary arithmetic in the form of the following scheme,

(PEM) $\exists n\psi(n) \vee \neg\exists n\psi(n)$,

where n ranges over natural numbers and $\psi(n)$ is a formula in the language of first-order arithmetic. The rejection of (PEM) is common to all varieties of constructivist thinking. The next example, on the other hand, is from Brouwer's particular brand of intuitionism:

(UCONT) Every real-valued function on the closed interval [0,1] is uniformly continuous.

Not all constructivists assert this, but classical mathematicians most definitely deny it. What is the nature of these disagreements? Consider the following possibilities.

2.1 Logical disagreement

The strongest way for A and B to differ over a claim φ is when A *asserts* φ and B *denies* it (or vice versa), where denial of φ is taken (following Frege) as assertion of the negation of φ. *If* (PEM) and (UCONT) express the same claims for both, then classical mathematicians and (Brouwerian) intuitionists disagree logically in this sense over (UCONT), but not over (PEM). The classical mathematician can easily produce real functions on the interval [0,1] that are not continuous; indeed introductory texts of analysis do that. As to (PEM) on the other hand, the negation of that principle is a logical contradiction, to everyone involved. The difference over (PEM) is of another kind.

2.2 Justificational disagreement

A weaker form of disagreement between A and B over φ is when A asserts it but B finds no justification for doing so (or vice versa). Intuitionists do not assert the negation of (PEM); they simply deny that there is any chance of ever finding a general justification for it.[2]

Logical and justificational disagreements presuppose that one and the same claim is involved. A weaker form of disagreement, that I will simply call *ideological*, does not require this.

2.3 Ideological disagreement

A may find important claims made by B incomprehensible, or incoherent, or lacking a solid foundation, and may therefore recommend (to her colleagues or students, for example) that one should not pursue the activity B engages in (or vice versa). Or she may understand them but find them unmotivated, uninteresting, or fruitless, and for this reason recommend against devoting time and energy to them. Here there need be no single claim over which A and B disagree (logically or justificationally); the disagreement is more practical and concerns what activities to promote.

Not any difference in attitude towards what is interesting or useful should count. If A finds topology exciting but analysis boring and B has the opposite attitude, this may have significant practical consequences – especially if A or B are in a position to influence colleagues, students, job committees, etc. – but should not count as ideological disagreement in the sense intended here. For ideological disagreement there should typically be one discipline – in this case mathematics – that each of A and B lays claims to, each finding that the other's activity is not *real* mathematics, or at least is in some way fundamentally flawed.

Looking at the literature, it certainly seems that more than ideological disagreement is involved. It is very common to *say* that theorems like (UCONT) *contradict* classical mathematics. For example, Brouwer, and Heyting following him,[3] emphatically speak of certain

[2] This latter denial is of a weaker sort than the previous one; otherwise it runs the risk of asserting the negation of (PEM) after all. If *justification* is the same as *proof*, and if asserting φ entails asserting that there is a proof of φ, then denying that there is any justification for (PEM) amounts to asserting its negation. Rather, the intuitionist notes the absence of any positive justification for (PEM) and the intuitive unlikelihood of ever finding one, without thereby purporting to be able to prove that no such justification exists.

[3] For example, Heyting (1971), p. 121 ff.

classical theorems as being contradictory, which means that the (intuitionistic) negation of such a theorem is provable.

3. Classical pluralism

We can be quite sure, however, that claims like the one just mentioned are not what they seem to be at first sight. For the claim, by someone who asserts or discusses (UCONT), say, is never that classical mathematics is *inconsistent*, only that it is *wrong*.[4]

It seems clear what is going on. Even though the actual terms used in (UCONT) are the same as the ones used by classical mathematicians, their extensions are different. For example, 'real number' and 'function' stand for quite different things in the intuitionistic context. Similarly, the logical symbols appearing in (PEM) have different meanings for the two parties involved. So Brouwer's claim is really that *if* the mathematical terms are used in the 'correct' – i.e., the intuitionistic, not the classical – way, then some sentences used to express classical theorems will become false under the intuitionistic reading.

This remark should be uncontroversial. In fact, it would be utterly trivial were it not for the fact that intuitionists make a point of using the classical language of mathematics, and sometimes of saying that certain classical theorems are simply *false*. In any case, the remark opens the way for the conciliatory methodological stance that I am going to call *classical pluralism*.

On this view, intuitionistic mathematics is a branch of mathematics, distinguished both by its restriction to certain constructive methods and, in some cases, its focus on certain mathematical objects not paid attention to by classical mathematicians. In elementary *arithmetic*, the objects studied are the same, i.e., the usual natural numbers with their usual operations, but the methods are more restricted. This is reflected in the fact that intuitionistic first-order arithmetic – Heyting Arithmetic (HA) – is a subsystem of classical first-order Peano Arithmetic (PA). Intuitionistic *analysis* (or second-order arithmetic), on the other

[4] Intuitionists tend to claim that classical mathematics runs a great *risk* of being inconsistent (as witnessed by the discovery of various paradoxes), because its use of illegitimate methods. But that is very far from claiming that it actually *is* inconsistent, i.e., that contradictions can be derived *inside* classical mathematics.

hand, studies in addition a different range of objects and need not be a subsystem of classical analysis, as witnessed by (UCONT).

Thus, there are two main aspects of the dispute between classical and intuitionistic mathematics: the *objects* talked about and the *words* used. Let me indicate how classical pluralism views each of these.

3.1 Objects

That the view just outlined is *pluralistic* is obvious, and it is *classical* in that it takes classical mathematics for granted. But there are two ways in which it can view the *objects* of the intuitionistic branch of mathematics.

3.1.1 Strong classical pluralism

The *strong* version of classical pluralism sees intuitionists as restricting attention to subclasses of classical mathematical objects. *Real numbers* is a typical case. Real numbers can be defined as (equivalence classes of) Cauchy sequences of rational numbers, but the intuitionist does not allow all such sequences. The restriction is enforced either by the stricter intuitionist logic alone, or by explicit extra demands on sequences, for example, 'lawlike' sequences given by some algorithm, or Brouwer's so-called choice sequences. There has been much debate, still going on, over the correct intuitionistic notion of infinite sequence. But however defined, they are all – according to strong classical pluralism – *sequences*, in the classical set-theoretic sense. More generally, *functions* in the set-theoretic sense comprise those given by rules, which are the ones preferred by constructivists. So it is not in the least surprising that a sentence involving universal quantification over functions from real numbers to real numbers, as in (UCONT), expresses very different claims in classical and in intuitionistic mathematics.

It is interesting to look at a well-known exposé of constructivism in mathematics like Troelstra and van Dalen (1988) from this viewpoint. The authors explicitly state that they do not accept any particular intuitionist or constructivist view as normative for mathematics, but that they present constructive mathematics as "a legitimate part of mathematics, containing material which is mathematically interesting, regardless of any philosophical bias"

(p. 5). This is almost a statement of classical pluralism.[5] Further, their long discussion of choice sequences often gives the impression of being about a given domain of objects, **N** → **N**,[6] the issue being which subclasses of this domain are constructively relevant. It is hard to understand this in any other way than that **N** → **N** is the classical or set-theoretic domain of functions from **N** to **N**. But their position is not quite clear (to me) on this point. The authors often emphasize that intuitionism studies *new* mathematical objects, with choice sequences as the prime example, and that this is part of its characteristic flavor. This *can* be taken merely as the affirmation that choice sequences, in contrast with, say, lawlike sequences, are unfamiliar to classical mathematics, not that they are not sequences in **N** → **N**. But it could also mean something stronger.

For example, it could mean that the identity conditions for choice sequences are different from those of ordinary sequences and that they therefore are a different kind of objects.[7] This is best illustrated with the more familiar lawlike sequences: two such sequences are identical if their respective laws or algorithms are the same. Assuming for the sake of argument that we know the identity criteria for algorithms, this certainly refines standard identity in **N** → **N**.[8] The difference can be phrased in terms of *intensional* vs. *extensional* identity. However, even if lawlike sequences or choice sequences are intensional objects in this sense, that doesn't necessarily make them non-classical. They could simply be identified with *pairs* of sequences from **N** → **N** and some other objects (like algorithms, Turing machines, etc.). Only if there is no classical account of those other objects do we get a serious extension of the domain of classical mathematical objects.[9]

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[5] Except that the role of classical mathematics is not mentioned. Also, the quote indicates the view that mathematicians can leave the philosophical aspects of the dispute aside, and focus on the mathematics.

[6] **N** is the set of natural numbers. It is sufficient to discuss sequences of *natural* numbers, since the construction of rational numbers from natural numbers is fairly uncontroversial.

[7] Cf. the discussion of equality in Troelstra and van Dalen (1988), p. 839 ff.

[8] Of course nothing can strictly refine the identity relation. What I mean is that a new class of objects is introduced, such that the lawlike sequences in **N** → **N** correspond to equivalence classes of these under an equivalence relation *R* (which thus corresponds to standard identity in **N** → **N**), and that the chosen 'identity' relation for the new objects refines *R*.

[9] Pagin (1998), in a slightly different context, sketches how intuitionistic logic can be viewed from a classical perspective, not just in the mathematical domain. He suggests that a predicate symbol *P* has a subclass of its classical extension

3.1.2 Weak classical pluralism

Classical pluralism isn't committed to all intuitionistic mathematical objects being classical. The *weak* form of this methodological stance grants that intuitionism may study genuinely new objects, but they are still *bona fide* mathematical objects. It is by no means strange that new branches of mathematics discover or invent new kinds of objects. Even if these objects can be rendered in classical set theory, that might not be the preferred way to view them. This issue is of course not limited to choice sequences; other cases could be *proofs* as mathematical objects, or *dependent types* as in Martin-Löf's type theory.

Furthermore, only some (albeit central) parts of mathematics, such as number theory, analysis, and set theory, can with any justification be viewed as being *about* certain objects. But consider topology. Even though some basic examples of topological spaces come from the 'objectual' part of mathematics, and even though abstract topology sometimes gives new insights about these objects, topology cannot reasonably be said to be about them. It is rather a system of concepts, characterized by certain definitions and axioms, which is widely applicable but can also be studied for its own sake. Only if you think of topology as part of set theory, it seems to me, does it make any sense to talk of topology having its proper objects. But even so, it is about sets in rather diluted sense, just as all mathematics, on this view, is about sets, even number theory. That is a particular (classical) view on the *foundations* of mathematics, but it is not part of what I here call classical mathematics, and not a view presupposed in this note.[10]

For abstract parts of mathematics such as topology or group theory, classical pluralism is simply the view that, say, constructive topology is just another piece of mathematics (presumably another piece of topology), introducing some new concepts and distinctions often interestingly related to the old ones, and restricting attention to certain methods of proof. As

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as its intuitionistic extension: the intuitionistic extension of 'horse' is the class of objects which *verifiably* or *provably* are horses. This is fully consonant with strong classical pluralism.

[10] But isn't topology about topological spaces, and group theory about groups, just as number theory is about natural numbers? Only in a diluted sense. The former might be said to be about all structures satisfying their axioms, but these structures can be wildly different, and there is no privileged one. Even though first-order number theory has non-standard models, and even though any standard model is only determined up to isomorphism, the standard model, or at least its isomorphism type, *is* privileged, and is what number theory 'is about' in the sense used above.

long as one is not talking about specific mathematical objects, the distinction between strong and weak classical pluralism disappears.

3.1.3 Three perspectives

According to classical pluralism there is no disagreement between classical and intuitionistic mathematics: intuitionistic mathematics is just a special kind of mathematics. An interest in constructive methods hardly needs motivation. For one thing, one wants to prove results under as weak assumptions as possible, and proving something in HA rather than PA assumes less, in a precise sense. For another, constructive proofs often give more information than classical ones. A related fact is that constructive methods lend themselves naturally to *computation*.

In principle, it would seem that at least weak classical pluralism is a stance available to the intuitionist too. She could see her activity simply as part of mathematics, along with classical mathematics, and motivate her preference for constructive methods with aesthetic criteria, or personal taste, or applicability to computation. In this way disagreement would be shown to be only apparent, and the conflict would dissolve.

But with few exceptions (see section 4.2 below), that is not how intuitionists have seen the matter. Instead, the usual attitude is that parts of classical mathematics are incoherent and in need of reform. And it is no accident that intuitionists use the same symbols for the logical constants. They claim that something is wrong with the classical use of those symbols and that their alternative reading is better.

Three possible perspectives on mathematical activity are involved here. One is the perspective of the practicing classical mathematician, another the perspective of the intuitionist. The third perspective is that of the outside observer – say, a philosopher of science. Classical pluralism is available to the practicing classical mathematician as well as the outside observer. That most intuitionists don't agree with this description is not necessarily a problem from the observer perspective. The main thing is that in a very strong sense, what looked like a conflict isn't really one. Not only is there no single mathematical claim about which the two parties disagree (logically or justificationally), in spite of appearances to the contrary. (They don't even disagree justificationally about (PEM); on the contrary, the classical mathematician fully agrees that *on the intuitionistic reading*, (PEM) is not justified.) This much is common to many non-relativist 'dissolutions' of relativistic claims: the parties are seen to be really talking about different things and so their respective assertions can in principle be jointly true. But here the parties can actually be seen as talking about the *same* things, or at least things belonging to a common range of mathematical objects; they are making different but not opposing claims about those objects. That the intuitionist may not see things quite that way just shows – according to classical pluralism from the observer perspective – that she is somewhat mistaken about her own activity, not that there is anything problematic with that activity. And even if she persists in taking this attitude, the remaining disagreement is at most ideological.

But so far we have only discussed the objects talked about and the concepts dealt with. At least as important are issues of meaning and truth. Maybe it is here that the greatest difficulties for a coherent version of classical pluralism arise, and maybe that is why such a position seems so unattractive to the intuitionist.

3.2 Truth and meaning

The fact that HA is a subsystem of PA means that any theorem of HA is also a theorem of PA. But these are formal systems, hence not directly concerned with meaning, so what does such a fact really tell us? We know that the intuitionistic reading of a sentence in the standard language of first-order arithmetic is usually very different from its classical reading. Consider (PEM). In asserting an instance of (PEM) in classical mathematics one is just saying that either there is a number with the property expressed by $\psi(n)$ or there is no such number. This is something one is entitled to affirm on trivial logical grounds. For the intuitionist, on the other hand, (PEM) is only assertible if (roughly) either one has a way of producing a number n together with a proof of $\psi(n)$, or a way of finding a proof that the existence of such a number leads to contradiction. Obviously, one may have no grounds for asserting this at all.[11] It is a totally different claim, and the fact that it *looks* the same as the classical one is rather *misleading*. Now, clearly the same must hold, not only for disputed claims, but also for claims that are provable in HA (and hence in PA): the same logical symbols with the same respective meanings are used.

[11] Brouwer's example is when $\psi(n)$ is the statement "the n'th decimal in the expansion of π is 9 and is preceded by 12345678".

This is a trivial point, obvious from the standard intuitionistic explanation of the meanings of the usual logical constants, the so-called Brouwer-Heyting-Kolmogorov (BHK) account.[12] For example, asserting $\neg\psi$ in classical mathematics is just saying that ψ is not true, but in intuitionistic mathematics it is saying that ψ is contradictory. Likewise, asserting $\varphi \rightarrow \psi$ in classical mathematics is saying that if φ is true so is ψ, but in intuitionistic mathematics it involves being in possession of a method that transforms any (canonical) proof of φ into a proof of ψ. It hardly needs saying that these are very different claims.

Now, because the respective claims are so different, and even more so the respective approaches to meaning and truth, it might be thought that classical pluralism cannot give a sensible account of the dispute between classical and intuitionistic mathematics. However, I will argue that the divergence about truth and meaning is not an obstacle to this enterprise.

Briefly, although the corresponding classical concepts are indeed vastly different from the intuitionistic ones, what really matters in the present context is the notion of *assertion*, and here both camps have the same idea. Let me begin with truth.

3.2.1 Truth

In the modern accounts of intuitionism given by Dummett, Prawitz, and Martin-Löf meaning-theoretic considerations play a crucial role.[13] Essentially, truth is no longer the central semantic concept; it is an *epistemic* concept, definable in terms of the central notion of *verification* or *proof*. Classical truth, on the other hand, is correspondence-theoretic and totally unrelated to epistemic notions.

A digression: *Must* intuitionistic truth be epistemic? One can say that a Tarski style or model-theoretic semantics is alien to the spirit of intuitionism, but might it not still apply, to those mathematical objects that intuitionists recognize? The idea of mathematics describing a structure of objects, even if these objects are mental constructions, is a powerful one. But

[12] For example, Heyting (1971), p. 102 ff., Prawitz (1974), p 63–77, Troelstra and van Dalen (1988), p. 9. The point is trivial, but rarely stated in the literature in the case when both readings are provable. Occasionally, e.g., Heyting in his early papers, one takes care to introduce new symbols for intuitionistic logical constants, perhaps alongside the classical ones. Then of course the difference cannot be missed. But usually the same symbols are used, and then, it seems to me, there is a tendency to forget that, even for simple arithmetical theorems provable both classically and intuitionistically, the contents of the respective claims can be quite different.

[13] See, for example, Dummett (1980), Prawitz (1974), Martin-Löf (1984).

the answer seems to be no. This is especially clear in the context of arithmetic. Intuitively, it is hard to think of a difference between natural numbers from the classical and from the intuitionistic perspective. There are indeed models in the classical sense of HA, namely the topological models first introduced by Tarski and later refined by Beth and Kripke.[14] But it is a striking fact that all of these models, unless they collapse into the standard classical model and thus validate (PEM) etc., are *non-standard*, in the sense of containing non-standard numbers.[15] And non-standard numbers are not what anyone wants arithmetic to be about, so the conclusion is that, at least so far, a model-theoretic account of intuitionistic mathematics is not available.

Granted, then, that intuitionistic truth is epistemic and classical truth is not, does this basic fact complicate or perhaps even invalidate the picture propounded by classical pluralism? I think not.

3.2.2 Assertion

My reason for so thinking is that classical and intuitionist mathematicians alike agree on the following:

1. Mathematicians make assertions, expressed in mathematical language.
2. An assertion requires justification, i.e., proof.
3. Provable claims are true.[16]

This means that they have the same idea of what the basic mathematical activity is about. Certainly, the intuitionist (roughly) *identifies* truth with the existence of a proof; the classical mathematician most definitely does not. But this has no consequences for what they consider to be the basic activity, namely, *discovering proofs of interesting mathematical*

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[14] See van Dalen (1986), pp. 243 ff. for an overview of such models.

[15] Ibid., p. 319.

[16] This, by the way, distinguishes mathematics from other disciplines, where one can be justified in asserting something which turns out to be false. Justification is possession of evidence, but not necessarily conclusive evidence. But in mathematics there can never be a proof of something false, and no weaker kind if justification is in principle accepted.

theorems.[17] It has consequences for which kinds of proofs are accepted, and for the content of the theorems, not for the nature of the activity itself.

3.2.3 Meaning

Similar remarks apply to the intuitionistic concept of meaning. This concept is somewhat elusive, at least to the outside observer. Intuitionists express the meaning of the logical constants in terms of assertion conditions of sentences of the relevant form. These conditions are rather clear, but it is not completely obvious how to extract the meaning of a *sentence* ψ from them. Or take the Dummett-Prawitz idea that to *know the meaning* of ψ is to know what a canonical proof of ψ looks like. It does not follow from this that the meaning *consists* in such canonical proofs, or in the ability discern them.

But again, these issues are inconsequential here. For what interests us is precisely the *intuitionistically assertible* mathematical sentences, and we do have a pretty clear idea of which these are, regardless of the fine points about exactly what intuitionistic meaning and truth amount to.

Thus, I want to separate the possible philosophical implications of the intuitionistic view on truth and meaning from the mere description of mathematical activity. The differences about the former do not have great effects on the latter, since both parties agree about the nature of assertion (in mathematics). Although a classical mathematician may believe that a mathematical proposition ψ can be true even if we will never find a proof of it (indeed, even if there is no proof), he can never *assert* ψ without a proof. The point is philosophically interesting, but not important to daily mathematical practice. Only provable claims are assertible.[18]

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[17] This is oversimplified. Discovering or inventing concepts, and finding good notation, are also basic activities. But there seems to be no fundamental divergence about these activities either. Here I am focusing on the 'assertive' part of mathematical activity.

[18] Famously, Dummett has argued that, given certain constraints on the notion of meaning, the classical view of meaning as truth conditions fails, and with it the principle of *bivalence*, and furthermore that this linguistic observation has the metaphysical consequence that *realism* fails; see Dummett (1976). The argument is criticized in Pagin (1998), who claims that even if the meaning-theoretic argument is sound, there are no metaphysical conclusions to be drawn. I am avoiding the issue of realism in this note, as well as meaning-theoretic subtleties, maintaining that classical pluralism can be still defended. Presumably, this is only true up to a point.

But what of that very assertion: "ψ may be true even if we will never find a proof of it"? That is not itself a mathematical assertion, it is a claim *about* mathematics and mathematicians. Of course, mathematicians do well to sometimes reflect on their own activity, but the claim just made is not something an intuitionist necessarily should disagree with. If she can make sense of the classical notion of truth, she should agree that the claim just expresses a property of that notion. If she interprets "true" as (roughly) "provable", the claim becomes patently false, but it is no longer the claim originally made. Finally, if she finds the classical notion of truth incomprehensible or incoherent, there is no clear claim to agree or disagree with. At most, there is ideological disagreement.

3.2.4 Intuitionistic vs. classical claims I

We can now spell out in somewhat more detail – though still very roughly – how classical pluralism views intuitionistic mathematics. The basic idea is the following:

(C) Intuitionistically provable claims, when understood correctly, are simply true mathematical propositions. This is because, although intuitionists have stronger requirements on proofs, all intuitionistic proofs are also classical proofs.

The last part of (C) is the crucial one. Often this part is unproblematic, in particular when the relevant piece of mathematics can be adequately formalized, and the resulting intuitionistic system is a subsystem of the classical version. This is the case with first-order arithmetic, but also with intuitionistic approaches to analysis or topology that do not rely on choice sequences but merely drop some classical assumptions like (PEM), or the Powerset Axiom, or Zermelo's Axiom of Choice.[19] Then every formal proof in the intuitionistic system is also a classical proof, and from this it seems clear that every corresponding 'real' intuitionistic

[19] Again I am oversimplifying: it is not just a case of dropping some axioms but also of choosing the right definitions. One of Brouwer's reasons for introducing choice sequences was that pointwise quantification over the constructive version of the continuum led to problems. He thought the continuum had to be redefined in terms of choice sequences. An alternative modern approach is that of *pointfree* topology, such as the 'formal topology' introduced by Martin-Löf and developed by Sambin, Coquand and others; cf. Sambin (2003) for an overview. This does not need choice sequences. It introduces new and finer distinctions than the classical approach, and it refrains from the classical Axiom of Choice, but the logic is essentially ordinary intuitionistic logic, which is part of classical logic.

proof is, or at least can be transformed into, a classical proof. (Recall that intuitionistic proofs are *constructions*, hence not quite identical with proof trees or proof sequences in a formal system.)

Note that this is no conclusive argument for the claim that *all* intuitionistic proofs pertaining to the relevant piece of mathematics are also classical. Since Gödel's theorems we know that no particular formal system can capture all truths in a sufficiently strong piece of mathematics, and intuitionists particularly emphasize the open character of the mathematical enterprise: new ideas and methods of proof may always appear. So presumably there can be no conclusive argument for (C) even in the present easy case. Rather, (C) expresses a conviction obtained by reflection on the intuitionist idea of proof, and is strongly supported by available evidence. The reason I think it is an unproblematic part of classical pluralism is that this conviction is shared, as far as I know, by intuitionists and classical mathematicians alike.

3.2.5 Intuitionistic vs. classical claims II

Now consider the case when the intuitionistic system is not a subsystem of any classical system, as with analysis based on choice sequences. The idea is that (C) still holds, but the justification is more complex.

To fix ideas, consider (UCONT). Suppose the intuitionist asserts (UCONT), by means of a proof. The question, from a classical perspective, is: What is she asserting, and is the assertion justified?

Assume that a language for the relevant piece of mathematics is informally specified, containing the usual mathematical constants, predicate symbols, function symbols, as well as possible new intuitionistic symbols of these kinds, and the usual logical symbols. The intuitionist uses the same language but interprets it differently. But that interpretation – both the re-interpretation of some of the usual symbols and the interpretation of the specifically intuitionist symbols – can also be specified classically. After all, that is what accounts of constructive mathematics like Troelstra and van Dalen (1988) are doing, and that is how the intuitionistic content of the sentence (UCONT) is explained. What this amounts to is the assumption that the intuitionistic claim expressed by a mathematical sentence can be rendered by *another* sentence in the same language.

Replace all the primitive expressions E such as "real number", "function", "continuous", etc. by classical descriptions E^i of their intuitionistic extensions. This will introduce new complex expressions, but in the result all primitive constants, predicate expressions, and function expressions have their usual classical meaning. The logical symbols are not touched.

An example will make it a little clearer how one can go from E to E^i.

Example: Consider, for example, "real number".[20] We can start by replacing this by "equivalence class of Cauchy sequences of rationals", but the intuitionistic extension of the primitive terms occurring here also needs to be specified. In particular, "sequence" should be replaced by "choice sequence". Now it is not clear that choice sequences really have a classical 'definition'. After all, the notion is still under debate. But what we can do – and what most modern expositors do – is to proceed axiomatically. What is required of choice sequences for a theorem like (UCONT) to be intuitionistically provable, is that they are sequences satisfying certain properties. So one can introduce a new predicate symbol for them and list the relevant axioms as background. This is nothing strange but a standard mathematical practice.[21]

I have given little more than a hint of how to find E^i. But the claim made by classical pluralism is that, to the extent that accounts of intuitionistic mathematics are at all possible, some such translation has to be possible. Let us say that it transforms a sentence ψ into ψ^i.

The second step is then to see that $(\text{UCONT})^i$ is justified, i.e., that it is provable classically. I will not attempt a general demonstration of this. The details depend a lot on particular choices of formalization, and I have not seen an attempt to give a justification that is independent of such formal details. Perhaps that would be not at all trivial. But again, it seems to me that there is no disagreement on this point: intuitionistic proofs are (or can be turned

[20] Assuming for simplicity that "real number" is a primitive expression. If it is defined, the remarks apply instead to the primitive expressions occurring in that definition.

[21] If choice sequences are seen as a special kind ordinary sequences, further axioms should make that clear. If not, choice sequences are seen as 'defined' by the relevant axioms.

into) classical proofs. In other words, (C) is presumably a substantial claim, but there seems to be no disagreement about it.

Example, continued. Consider, for illustration, the following possibility: Suppose the notion of choice sequence is completely axiomatized, in the sense that all properties of choice sequences used in the intuitionistic proof of (UCONT) are given in some finite set of axioms. Similarly for other primitive notions needed. Collect these axioms in one sentence, AX.

(UCONT) may not contain a symbol "choice sequence" directly; rather, it contains, say, "real number", and AX includes a (standard) definition of "real number" in terms of Cauchy (choice) sequences. Suppose next that the proof of (UCONT) from AX can be formalized in something like intuitionistic predicate logic. And suppose also that the proof is *schematic*, in the sense that the non-logical symbols occurring in it are just symbols, which can be replaced (appropriately) without disturbing the proof. These assumptions are not unrealistic. Then replace "choice sequence" by the formula (choice sequence)^{i} which defines it classically, and similarly for other symbols.[22] The proof remains valid; it is now a proof of (UCONT)^{i} from AX^{i}. Moreover, it is a classical proof, since intuitionistic predicate logic is a subsystem of classical predicate logic.

Finally, suppose there are classical (presumably set-theoretic) axioms from which AX^{i}, or at least those parts of AX^{i} which are not mere definitions, can be derived. For example, these axioms would say that the sequences talked about in AX^{i} are 'real' set-theoretic sequences. Alternatively, if choice sequences are not thought of as classical sequences but merely as axiomatically introduced, the classical mathematician could be content to use AX^{i} as axioms. In any case, there would now be a classical proof of (UCONT)^{i}.

4. In conclusion

I hope to at least have made a case for what I called classical pluralism, a case that is classically acceptable and shows that faultless disagreement is not what occurs in the present conflict. But I have no illusions of having persuaded many intuitionists. Even if, as I have ar-

[22] As to "real number", it can probably be replaced by another *primitive* symbol "(real number)^{i}", since the definition of real numbers in terms of Cauchy sequences has the same form classically and intuitionistically.

gued, the philosophical differences concerning truth and meaning are less important than it might seem, I have simply taken classical mathematical practice for granted. And most intuitionists insist that parts of this practice is unjustified or incoherent.

I didn't set out, however, to convince the intuitionist, only to tell a coherent story for the outside observer. But now such an observer might complain that so far I have only told half the story. What if one sides with the intuitionists instead, and tries to see classical mathematics from their perspective? And if this is possible, what is the nature of disagreement between that picture and classical pluralism? I can only sketch an answer here.

4.1 Negative translations

Those intuitionists – the vast majority – who reject parts of classical mathematics as meaningless or not coherent, can of course not make sense of those parts, at least not without reinterpreting them. So the situation is not symmetric: classical mathematicians do not find intuitionistic mathematics meaningless, only somewhat special, but the inverse is not true.

However, intuitionists have indeed been interested in 'making sense' of classical mathematics by *reinterpreting* it. A common view seems to be that you can understand much of the 'illicit' parts of classical mathematics, not in the classical way but via the so-called *negative translations*. For example, if you understand (PEM) rather as the classically but not intuitionistically equivalent

$$\neg\neg(\exists n\psi(n) \vee \neg\exists n\psi(n)),$$

it becomes indeed intuitionistically valid. And as Gödel showed, all of PA can be faithfully translated in to HA in the sense that a sentence is provable in PA if and only if its translation is provable in HA. Similar translations have been given for other pieces of classical mathematics as well.

It would be interesting to investigate this idea, in particular what exactly such translations tell us. Reflection on the very notion of *translation* would be required.[23] I am somewhat doubtful that these mappings really qualify as translations in the full sense, but that argument has to be left for another occasion.

[23] Some efforts in this direction can be found in chapter 11 of Peters and Westerståhl (forthcoming).

4.2 Constructive pluralism

The symmetric intuitionist position with respect to classical pluralism would be one that finds classical mathematics meaningful, but best seen as part of constructive mathematics; let us call this *constructive pluralism*. One of the very few constructivists who has expressed a similar view in writing is Giovanni Sambin; see in particular Sambin (2003), which in addition to presenting formal topology also has some philosophical discussion.

But how can a system which I have described as 'more restrictive' also be more general? Actually, these terms are ambiguous. Let IS and CS be an intuitionistic and a classical system, respectively, for asserting mathematical claims, i.e., for proving theorems. CS can be more general in the sense that everything provable in IS is also provable in CS but not *vice versa*. But *at the same time*, IS could be more general CS in that CS is the result of setting some parameters in IS in a certain way, or obliterating some distinctions that IS makes. The simplest example is intuitionistic vs. classical *logic*: the addition of the principle of excluded middle, which gives classical logic, can be seen as destroying a distinction that the intuitionistic system allows, or identifying things (in this case, falsity with not being true) that it allows to be different. It *allows* them to be different but does not *require* them to be, and in this sense classical logic becomes a special case.

Sambin argues that the same is true for topology and, I presume, for mathematics in general. His formal topology makes more distinctions and introduces new concepts, but it does not rule out classical topology, which instead becomes a special case. For example, the basic notions of *open* and *closed* sets are still there, but classically closed sets are simply defined as complements of open sets. According to the constructivist, this only 'works' under classical logic. Instead, you can introduce closed sets separately, and discover that there is indeed an interesting duality between open and closed sets, but not to the point of one being definable in terms of the other. A new distinction is made, and classical topology falls out as a special case.

The classical pluralist thinks you should do constructive mathematics if you find it more useful for some purpose, like computation, or simply if you find it more aesthetically pleasing. The constructive pluralist thinks you should do classical mathematics if you care more about simplicity and a certain formal elegance than about computation, perhaps because you find *that* more aesthetically pleasing. Aesthetic ideals vary, but Sambin maintains that by actually doing constructive mathematics (in the right way), one will see that its reputation for being boring is misplaced, and that it contains true mathematical beauty.

So what is the nature of the disagreement between classical and constructive pluralism? There is certainly no logical or justificational disagreement, but the remaining category I used, 'ideological disagreement', does not fit very well either. I leave for another occasion, or to the interested reader, the formulation of an appropriate methodological stance towards *this* dispute.

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